Econ 802: Concavity, Second Order Conditions, And So On

October 5, 2005

Here is a summary of various conditions for profit maximization and cost minimization. In each column, conditions are ranked from weakest to strongest. Thus 4A implies 3A, 3A implies 2A, and 2A implies 1A. However, 1A does not imply 2A, 2A does not imply 3A, and 3A does not imply 4A. The same is true for the right column. Also, 1A implies 1B and so on, but the converse is not true (1B does not imply 1A).

Profit maximization	Cost minimization
1A. The Hessian of the production function is neg semi def at a point x* satisfying the FOC.	1B. The Hessian of the production function is neg semi def in directions orthogonal to the gradient vector at a point x* satisfying the FOC.
2A. Global concavity of the production function. This is the same as having a Y set that is convex along its upper boundary. It implies that the Hessian is neg semi def at all x.	2B. Global quasi-concavity of the production function. This is the same as having convex V(y) sets. It implies that the Hessian of the production function is neg semi def in directions orthogonal to the gradient vector at all x.
3A. Strict global concavity of the production function. This is the same as having a Y set that is strictly convex along its upper boundary.	3B. Strict global quasi-concavity of the production function. This is the same as having strictly convex V(y) sets.
4A. The Hessian of the production function is negative definite at all x.	4B. The Hessian of the production function is negative definite in directions orthogonal to the gradient vector at all x.

1A is a necessary SOC but it is not sufficient (if x* is a solution then 1A must hold, but 1A could hold even if x* is not a solution). A sufficient SOC for profit max is that the Hessian be negative definite at x*. However, this is a local condition, not a global one, so it only ensures that x* solves the problem locally. The remaining conditions (2A, 3A, 4A) are all global statements. Each ensures that if x* satisfies FOC, it is a global solution (so if any of these is true, don't bother checking the SOC at x*). Condition 3A ensures uniqueness of a solution. Condition 4A is even better because it ensures that the solution can be treated as an implicit function of the prices. If we have neg def in 1A, this is also true, but only locally (we can use the implicit function in a neighborhood of the current prices but maybe not elsewhere). Similar things are true in the right column.